DIFFERENT DEFINITIONS OF THE RATE OF CHANGE OF A TENSOR*

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Tensors can be regarded as entities linked to definite particles in a moving continuum. It is, further, possible to introduce, in many senses, individual derivatives with respect to time i.e. the rates of change of tensors.

It is easy to formulate a complete theory of differentiation of tensors of arbitrary order with respect to a scalar parameter if use is made of known techniques of operating on tensors [1,2] regarded as invariant entities and represented in the form of symbolic sums

$$T = T^{\alpha\,\cdot\,\gamma}_{\cdot\,\beta\,\cdot}\,\mathfrak{s}_\alpha \mathfrak{s}^\beta \mathfrak{s}_\gamma \tag{1}$$

Here $\mathfrak{s}_\alpha$ and $\mathfrak{s}^\alpha$ (α = 1, 2, 3) are the covariant and contravariant base vectors of the coordinate system. These vectors can be functions of position in space and of time t.

In a manner similar to that which is used when different velocity vectors are employed in the mechanics of a rigid body, the study of the motion of a deformable continuum can be conducted with the aid of suitable rates of change of tensors which can be defined in various ways.

In his lecture, W. Prager introduced intuitive considerations on four different forms of the stress rate tensor in Cartesian coordinates which

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* The present paper represents a more detailed development of the author's contribution to the discussion on W. Prager's lecture "An elementary discussion of definitions of stress rate" which he gave at the First All-Union Congress on Theoretical and Applied Mechanics in January 1960 in Moscow.

were proposed, respectively, by Jaumann [3], Cotter and Rivlin [4], Oldroyd [5], and Truesdell [6].

In the theory of plasticity and visco-elasticity, as well as in other cases, Prager favored Jaumann's definition because it excludes the effect of rotation of the neighborhood of the particle considered, and because the derivatives with respect to time of the invariants of the stress tensor vanish simultaneously with the stress rate.

In what follows we shall establish the inner connection between the above rates of change of tensors in arbitrary, curvilinear coordinate systems. We shall introduce additional rate-of-change tensors which have an essential significance, and we shall show that Prager's argument on the basis of which he demonstrated the superiority of Jaumann's representation is insufficient to define the concept of the stress rate. Furthermore, we shall introduce supplementary concepts which will allow us to establish, on the hand of examples, the rules for the application of the derivatives of a tensor with respect to a parameter in different senses.

Let us consider curvilinear systems of coordinates in which the "juggling" of indices is accomplished with the aid of a fundamental metric tensor

$$G = g_{ij}\text{э}^i\text{э}^j = g^{ij}\text{э}_i\text{э}_j = \delta_j{}^i\text{э}_i\text{э}^j$$

where the square of the element of length ds is given by the formula

$$ds^2 = g_{\alpha\beta}dx^\alpha\, dx^\beta, \qquad g_{\alpha\beta} = (\text{э}_\alpha, \text{э}_\beta)$$

We shall consider some moving medium which fills the space in a continuous way. Let the particles of the continuum be identified with the aid of a Lagrangean system of coordinates ξ^1, ξ^2, ξ^3, defined in a moving curvilinear system of coordinates, that is linked to the medium and possessing base vectors $\hat{\text{э}}_i$ and $\hat{\text{э}}^i$ $(i = 1,\ 2,\ 3)$. The quantities ξ^1, ξ^2, ξ^3, can be regarded as coordinates in a fixed system with the base vectors $\mathring{\text{э}}_{i'}$ and $\hat{\text{э}}^i$ which coincide with the moving system э_i and $\hat{\text{э}}^i$ at some initial instant of time t_0.

We shall denote by x^1, x^2, x^3 the coordinates of the points of space with bases э^i and э_i in the reference system with respect to which the motion of the moving continuum is determined.

The law of motion is represented by functions of the form

$$x^i = x^i(\xi', \xi^2, \xi^3, t)$$

We shall denote by $\mathbf{r}$ the radius-vector of the points of the space, and after determining the base vectors we have

$$d\mathbf{r} = dx^{\alpha}\mathbf{э}_{\alpha}, \qquad d\mathbf{r} = d\xi^{\alpha}\hat{\mathbf{э}}_{\alpha}, \qquad d\mathbf{r}_0 = d\xi^{\alpha}\mathring{\mathbf{э}}_{\alpha} \tag{2}$$

The velocities of the particles are determined by the equations

$$\mathbf{V} = \left(\frac{d\mathbf{r}}{dt}\right)_{\xi^i} = \frac{\partial x^{\alpha}}{\partial t}\mathbf{э}_{\alpha} = v^{\alpha}\mathbf{э}_{\alpha} = \hat{v}^{\alpha}\hat{\mathbf{э}}_{\alpha} \tag{3}$$

It is evident that to every tensor defined in the deformed space whose metric is

$$ds^2 = \hat{g}_{\alpha\beta}d\xi^{\alpha}\,d\xi^{\beta} = g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta}$$

there correspond several different tensors, with different components defined in the space of initial states whose metric is

$$ds_0{}^2 = \mathring{g}_{\alpha\beta}d\xi^{\alpha}\,d\xi^{\beta}, \qquad \mathring{g}_{\alpha\beta} = (\mathring{\mathbf{э}}_{\alpha}\cdot\mathring{\mathbf{э}}_{\beta})$$

Different tensors will be obtained for different fixed systems of covariant and contravariant indices for which the equality of components is achieved. The components of corresponding tensors in different spaces in the second system of indices, different from the fixed one, turn out to be different. For example

$$\boldsymbol{T} = \hat{T}^{\alpha\cdot\gamma}_{\cdot\beta\cdot}\hat{\mathbf{э}}_{\alpha}\hat{\mathbf{э}}^{\beta}\hat{\mathbf{э}}_{\gamma}, \qquad \mathring{\boldsymbol{T}} = \mathring{T}^{\alpha\cdot\gamma}_{\cdot\beta\cdot}\mathring{\mathbf{э}}_{\alpha}\mathring{\mathbf{э}}^{\beta}\mathring{\mathbf{э}}_{\gamma}$$

where

$$\hat{T}^{\alpha\cdot\gamma}_{\cdot\beta\cdot} = \mathring{T}^{\alpha\cdot\gamma}_{\cdot\beta\cdot}, \qquad \hat{\boldsymbol{T}}_{\alpha\beta\cdot}{}^{\gamma} = \hat{g}_{\alpha\omega}\hat{T}^{\omega\cdot\gamma}_{\cdot\beta\cdot} \neq \mathring{T}_{\alpha\beta\cdot}{}^{\gamma} = \mathring{g}_{\alpha\omega}\mathring{T}^{\omega\cdot\gamma}_{\cdot\beta\cdot}$$

To one tensor $\mathring{\mathbf{T}}$ there correspond several tensors in the deformed space.

The differentiation of tensors of arbitrary order with the aid of the representations in (1) reduces itself to the differentiation of the components and base vectors and is completely analogous to the problem of the differentiation of vectors.

From (2) and (3) it is easy to derive the formulas

$$\begin{aligned} &\frac{d\hat{\mathbf{э}}_i}{dt} = \nabla_i v^{\alpha}\hat{\mathbf{э}}_{\alpha}, \qquad &&\frac{d\hat{\mathbf{э}}^i}{dt} = -\nabla_{\beta}v^i\hat{\mathbf{э}}^{\beta} \\ &\frac{d\mathbf{э}_i}{dt} = v^{\lambda}\Gamma_{\lambda i}{}^{\alpha}\mathbf{э}_{\alpha}, \qquad &&\frac{d\mathbf{э}^i}{dt} = -v^{\lambda}\Gamma_{\lambda\beta}{}^{i}\mathbf{э}^{\beta} \end{aligned} \tag{4}$$

where

$$\Gamma_{jk}{}^{i} = \frac{1}{2}g^{i\alpha}\left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^{\alpha}}\right)$$

We shall consider, additionally, a Cartesian base $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ which rotates with respect to the reference system $\mathfrak{s}_i$ with a given angular velocity $\Omega = \Omega_\beta \mathbf{i}_\beta$. For the derivatives $d\mathbf{i}_\alpha/dt$ we can write

$$\frac{d\mathbf{i}_\alpha}{dt} = \Omega \times \mathbf{i}_\alpha = \Omega_{\beta\alpha}\mathbf{i}_\beta \tag{5}$$

Here

$$\Omega_{21} = -\Omega_{12} = +\Omega_3, \quad \Omega_{31} = -\Omega_{13} = -\Omega_2, \quad \Omega_{23} = -\Omega_{32} = -\Omega_1$$
$$\Omega_{11} = \Omega_{22} = \Omega_{33} = 0, \qquad \Omega^{i\cdot}_{\cdot j} = \Omega_{ij}$$

Any tensor $\boldsymbol{H}$ of the second order can be represented in the following alternative forms

$$\boldsymbol{H} = h_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = h^{i\cdot}_{\cdot j}\hat{\mathfrak{s}}_i\hat{\mathfrak{s}}^j = h_{i\cdot}^{\cdot j}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}_j = h^{ij}\hat{\mathfrak{s}}_i\hat{\mathfrak{s}}_j = h'^{ij}\mathfrak{s}_i\mathfrak{s}_j = h^{*ij}\mathbf{i}_i\mathbf{i}_j \tag{6}$$

The systems $\mathfrak{s}_i$ and $\mathring{\mathfrak{s}}_i$ can be regarded as being Cartesian. The system $\hat{\mathfrak{s}}_i$, considered in the course of time, is intrinsically curvilinear. If we set $t = t_0$, we can assume that all three systems coincide at the given instant (they can be curvilinear or Cartesian).

To tensor $\boldsymbol{H}$ there correspond the different tensors

$$\mathring{\boldsymbol{H}}_1 = h_{ij}\mathring{\mathfrak{s}}^i\mathring{\mathfrak{s}}^j, \qquad \mathring{\boldsymbol{H}}_2 = h^{i\cdot}_{\cdot j}\mathring{\mathfrak{s}}_i\mathring{\mathfrak{s}}^j, \qquad \mathring{\boldsymbol{H}}_3 = h_{i\cdot}^{\cdot j}\mathring{\mathfrak{s}}^i\mathring{\mathfrak{s}}_j, \qquad \mathring{\boldsymbol{H}}_4 = h^{ij}\mathring{\mathfrak{s}}_i\mathring{\mathfrak{s}}_j \tag{7}$$

We shall regard all components of the tensors $\boldsymbol{H}$ and $\mathring{\boldsymbol{H}}_i$ defined in the different, alternative ways as functions of the Lagrangean variables ξ^i and the time t. From (7) and (6), in view of (4) and (5), we see that the different derivatives with respect to time t are given by the equations

$$\mathring{\mathbf{V}}_1 = \frac{d\mathring{\boldsymbol{H}}_1}{dt} = \frac{dh_{ij}}{dt}\mathring{\mathfrak{s}}^i\mathring{\mathfrak{s}}^j, \qquad \mathring{\mathbf{V}}_2 = \frac{dh^{i\cdot}_{\cdot j}}{dt}\mathring{\mathfrak{s}}_i\mathring{\mathfrak{s}}^j, \qquad \mathring{\mathbf{V}}_3 = \frac{dh^{ij}}{dt}\mathring{\mathfrak{s}}_i\mathring{\mathfrak{s}}_j \tag{8}$$

$$\begin{aligned}\frac{d\boldsymbol{H}}{dt} &= \left(\frac{dh_{ij}}{dt} - h_{\omega j}\nabla_i v^\omega - h_{i\omega}\nabla_j v^\omega\right)\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \left(\frac{dh^{i\cdot}_{\cdot j}}{dt} + h_j^{\cdot\omega}\nabla_\omega v^i - h^{i\cdot}_{\cdot\omega}\nabla_j v^\omega\right)\hat{\mathfrak{s}}_i\hat{\mathfrak{s}}^j = \\ &= \left(\frac{dh^{ij}}{dt} + h^{\omega j}\nabla_\omega v^i + h^{i\omega}\nabla_\omega v^j\right)\hat{\mathfrak{s}}_i\hat{\mathfrak{s}}_j = \left(\frac{dh'^{ij}}{dt} + v'^\lambda\Gamma'^{\;i}_{\lambda\omega}h^{\omega j} + v'^\lambda\Gamma'^{\;j}_{\lambda\omega}h^{i\omega}\right)\mathfrak{s}_i\mathfrak{s}_j = \\ &= \left(\frac{dh^{*ij}}{dt} + h^{*\omega j}\Omega^{i\cdot}_{\cdot\omega} + h^{*i\omega}\Omega^{j\cdot}_{\cdot\omega}\right)\mathbf{i}_i\mathbf{i}_j\end{aligned} \tag{9}$$

The analogous expression for the derivatives of $h_{i\cdot}^{\cdot j}$ has been omitted here for the sake of brevity.

Taking into account that at a given instant of time the bases $\mathfrak{s}_i$ and $\hat{\mathfrak{s}}_i$ coincide, we obtain the following formulas which are satisfied in curvilinear systems of coordinates:

$$\frac{dh_{ij}}{dt} = \frac{dh'_{ij}}{dt} + h_{\omega j}\frac{\partial v^{\omega}}{\partial x^{i}} + h_{i\omega}\frac{\partial v^{\omega}}{\partial x^{j}} \tag{10}$$

$$\frac{dh^{ij}}{dt} = \frac{dh'^{ij}}{dt} - h^{\omega j}\frac{\partial v^{i}}{\partial x^{\omega}} - h^{i\omega}\frac{\partial v^{i}}{\partial x^{\omega}} \tag{11}$$

$$\frac{dh^{i\cdot}_{\cdot j}}{dt} = \frac{dh'^{i\cdot}_{\cdot j}}{dt} - h^{\omega\cdot}_{\cdot j}\frac{\partial v^{i}}{\partial x^{\omega}} + h^{i\cdot}_{\cdot\omega}\frac{\partial v^{\omega}}{\partial x^{j}} \tag{12}$$

If h_{ij} is a symmetric tensor, then tensors (10) and (11) are also symmetric, but tensor (12) and the tensor for $dh_{i\cdot}^{\cdot j}/dt$ are, in general, asymmetric.

Taking into account that the system $\mathbf{i}_i$ coincides with the system $\mathfrak{d}_i$, we can derive one more formula in a Cartesian system which is applicable irrespective of the method of arranging indices

$$\frac{dh^{*ij}}{dt} = \frac{dh'^{ij}}{dt} - h'^{\alpha j}\Omega^{\,i\cdot}_{\cdot\alpha} - h'^{i\alpha}\Omega_{\cdot}^{\,j}{}_{\alpha} \tag{13}$$

The left-hand sides of (10), (11), and (12) contain the components of different tensors; to these tensors in the space of intial states, there correspond different tensors $\overset{\circ}{V}_i$ and, correspondingly, different tensors V_i in the deformed space. The quantities dh'_{ij}/dt and others are not components of tensors; they can be regarded as components of tensors only in a Cartesian system of coordinates.

The formulas in Equations (10) and (11) have been discussed in Prager's lecture. Here, the formula in (11) corresponds to the one introduced by Oldroyd [4], and the formula in [10], to that introduced by Cotter and Rivlin [3]. The derivatives for $dh^{i\cdot}_{\cdot j}/dt$ and $dh_{i\cdot}^{\cdot j}/dt$ were not mentioned in Prager's paper.

In nonlinear elasticity, the ratios of the components of the stress tensor and the density $p^{ij}/\rho = \sigma^{ij}$ turn out to be thermodynamic quantities which possess a potential. The derivative (11) taken for σ^{ij} after the substitution of p^{ij}/ρ for σ^{ij} and after multiplication by ρ turns out to be the derivative* of p^{ij} in Truesdell's sense [6]. Along with this derivative, it is possible to consider other analogous derivatives which are consequences of Equations (10), (12) and (13).

When generalizing the theory of small elastic and plastic deformations to include the case of finite deformations, it is probably fruitful to

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* This fact was clarified by Aspirant V.D. Bonder.

eliminate the tensor σ^{ij} in favor of the tensor p^{ij} in the corresponding formulas.

Let us consider the tensor of finite deformations ϵ and the tensor e of the velocities of deformation. We then have by definition

$$\varepsilon = \varepsilon_{ij}\hat{\mathfrak{z}}^i\hat{\mathfrak{z}}^j = \varepsilon_{ij}'\mathfrak{z}^i\mathfrak{z}^j \qquad \left(\varepsilon_{ij} = \frac{1}{2}\left(\hat{g}_{ij} - \mathring{g}_{ij}\right)\right) \tag{14}$$

$$e = e_{ij}\hat{\mathfrak{z}}^i\hat{\mathfrak{z}}^j = e_{ij}'\mathfrak{z}^i\mathfrak{z}^j \qquad \left(e_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right)\right) \tag{15}$$

At the instant when the bases $\hat{\mathfrak{z}}^i$ and $\mathfrak{z}^i$ coincide, we have

$$\varepsilon_{ij} = \varepsilon_{ij}', \qquad e_{ij} = e_{ij}', \qquad \varepsilon_j^{i} = \varepsilon_j'^{i} \text{ etc.}$$

Taking this into account, the definitions in (14), (15), together with Equations (10), (11), and (12), yield

$$\begin{aligned} e \sim \frac{d\varepsilon_{ij}}{dt} &= e_{ij} = \frac{d\varepsilon_{ij}'}{dt} + \varepsilon_{\omega j}\frac{\partial v^{\omega}}{\partial x^{\alpha}} + \varepsilon_{i\omega}\frac{\partial v_{\omega}}{\partial x^{j}} \\ e' \sim \frac{d\varepsilon^{i\cdot}_{\cdot j}}{dt} &= e^{i\cdot}_{\cdot\alpha}\left(\delta^{\alpha\cdot}_{\cdot j} - 2\varepsilon^{\alpha\cdot}_{\cdot j}\right) = \frac{d\varepsilon_j'^{i}}{dt} - \varepsilon^{\alpha\cdot}_{\cdot j}\frac{\partial v^i}{\partial x^{\alpha}} + \varepsilon^{i\cdot}_{\cdot\alpha}\frac{\partial v^{\alpha}}{\partial x^j} \\ e'' \sim \frac{d\varepsilon^{ij}}{dt} &= e^{\alpha\beta}\left(\delta_{\alpha}{}^{i}\delta_{\beta}{}^{j} - 2\delta_{\alpha}{}^{i}\varepsilon_{\beta}{}^{j} - 2\delta^{j\cdot}_{\cdot\beta}\varepsilon^{i\cdot}_{\cdot\alpha}\right) = \frac{d\varepsilon'^{ij}}{dt} - \varepsilon^{\omega j}\frac{\partial v^i}{\partial x^{\omega}} - \varepsilon^{i\omega}\frac{\partial v^j}{\partial x^{\omega}} \end{aligned} \tag{16}$$

These systems of derivatives represent essentially different tensors e, e', e'' which vanish when the system moves as a rigid body. When the deformations are infinitesimal, we have $e = e' = e''$.

The systems of derivatives

$$\frac{d\varepsilon_{ij}'}{dt}, \qquad \frac{d\varepsilon'^{i\cdot}_{\cdot j}}{dt}, \qquad \frac{d\varepsilon'^{ij}}{dt}$$

in curvilinear coordinates are not tensors; they are different from zero when the system moves as a rigid body.

If the base vectors $\mathbf{i}_i$ rotate with the angular velocity of the principal axes of tensor e, equal to zero, then in any system of coordinates we can write

$$\Omega_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_{\alpha}}{\partial y^{\beta}} - \frac{\partial v_{\beta}}{\partial y^{\alpha}}\right) \tag{17}$$

In this case Equation (13) determines the components of the derivative tensor introduced by Jaumann.

Taking into account that $\hat{\mathfrak{z}}_i = \mathfrak{z}_i = \mathbf{i}_i$ as well as (17), Equations (12) and (13) lead to

$$\frac{dh^{i\cdot}_{\cdot j}}{dt}=\frac{dh^{*i\cdot}_{\cdot j}}{dt}-h_{\alpha j}e_{\alpha i}+h_{i\alpha}e_{\alpha j}=\frac{dh^{*i\cdot}_{\cdot j}}{dt}+A_{ij} \tag{18}$$

It is easy to see that if tensor h_{ij} is symmetric, then tensor $dh^{*i\cdot}_{\cdot j}/dt$ is also symmetric and that tensor A_{ij} is anti-symmetric.

Next we consider the case when H is a symmetric tensor. It is evident that $A_{ij}=0$ if the principal axes of tensors H and e coincide. If P denotes the stress tensor, then in the case of a nonlinear elastic, isotropic body, the tensors P and e possess, in general, different principal axes, and in this case $A_{ij}\neq 0$. If $H=\epsilon$, we may write

$$\frac{d\varepsilon^i_j}{dt}=\left(\frac{d\varepsilon^i_j}{dt}\right)^*+\varepsilon^{i\cdot}_{\cdot\alpha}e^{\alpha\cdot}_{\cdot j}-\varepsilon^{\alpha\cdot}_{\cdot j}e^{i\cdot}_{\cdot\alpha} \tag{19}$$

The systems of invariants for the tensors H and $\overset{\circ}{H}_2$ are identical but, in general, they differ from the invariants of tensors $\overset{\circ}{H}_1$ and $\overset{\circ}{H}_4$. For example, the second invariants for tensors $\overset{\circ}{H}$, and H can be written

$$\overset{\circ}{J}_2=\overset{\circ}{h}{}^{\alpha\cdot}_{\cdot\beta}\overset{\circ}{h}{}^{\beta\cdot}_{\cdot\alpha}=\overset{\circ}{g}_{\alpha\lambda}\overset{\circ}{g}_{\beta\mu}h^{\lambda\beta}h^{\mu\alpha}$$

$$\hat{J}_2=h^{\alpha\cdot}_{\cdot\beta}h^{\beta\cdot}_{\cdot\alpha}=\hat{g}_{\alpha\lambda}\hat{g}_{\beta\mu}h^{\lambda\beta}h^{\mu\alpha}=4\varepsilon_{\lambda\alpha}\varepsilon_{\beta\mu}h^{\lambda\beta}h^{\mu\alpha}+4\varepsilon_{\alpha\lambda}\check{g}_{\beta\mu}h^{\lambda\beta}h^{\mu\alpha}+\check{J}_2$$

If the derivatives dh^{ij}/dt, determined by Equations (11), are equal to zero, then $d\overset{\circ}{J}_2/dt=0$, and, in general, the magnitude of dJ_2/dt differs from zero.

It is evident that the derivatives of any invariants of tensor $\overset{\circ}{H}_1$ vanish simultaneously with dh^{ij}/dt, determined by Equation (11), and that the derivatives of the invariants of $\overset{\circ}{H}_4$, vanish simultaneously with the derivatives of dh_{ij}/dt, determined by Equation (10).

It is easy to see that the derivatives of any invariants of tensor $\overset{\circ}{H}_2$ in the space of initial states and of tensor H in the deformed space vanish when $dh^{i\cdot}_{\cdot j}/dt=0$ in accordance with Equation (12), or when $dh^{*i\cdot}_{\cdot j}/dt=0$ in accordance with Equation (13), assuming the validity of Equation (18). The formulas for the elementary increments from any correspondingly equal invariants of tensors $\overset{\circ}{H}_2$ and H are identical owing to the increments $dh^{i\cdot}_{\cdot j}$ and $dh^{*i\cdot}_{\cdot j}$ which are connected by Equation (18). The preceding argument is valid also in the case when A_{ij} is an arbitrary antisymmetric tensor.

When physical laws are formulated, the mechanical processes undergone by a particle are described from Lagrange's point of view. The formulation of equations of motion with respect to reference systems can be carried out in fixed coordinate systems with respect to the initial space or in moving Lagrangean systems in the deformed space. In either case it is possible to introduce rates-of-change of tensors (for the strain tensors, for the stress tensors, for the ratio of the stress tensor to density, etc.) in the senses defined by Equations (10), (11) and (12).

The use of time-derivatives, as introduced by Jaumann, can be convenient because in this case the effect of the rotation is excluded, and the corresponding rate of change of a symmetric tensor turns out to be a symmetric tensor.

The effects, connected with the existence of different derivatives with respect to time of tensors taken with respect to different, previously defined, vector bases may turn out to be purposeful in the theory of motion of continua in the presence of infinitely small deformations in cases when displacements and rotations of axes of deformation are finite.

BIBLIOGRAPHY

1. Sedov, L.I., *Osnovy nelineinoi mekhaniki sploshnoi sredy (Fundamentals of nonlinear mechanics of continua)*. Izd. Moskov. Gos. Univ. 1959 (Notes of lectures given by the author at the Moscow State University in 1956-1959).

2. Lagally, M., *Vorlesungen über Vektor-Rechnung*. Leipzig, 1928.

3. Jaumann, G., *Grundlagen der Bewegungslehre*. Leipzig, 1905. See also *Sitz. Ber. Akad. Wiss. Wien*. IIa Vol. 120, p. 385, 1911.

4. Cotter, B.A. and Rivlin, R.S., Tensors associated with time-dependent stress. *Q.ly Appl. Math.* Vol. 13, p. 177, 1955.

5. Oldroyd, J.G., On the formulation of rheogical equations of state. *Proc. Roy. Soc.* A, Vol. 200, p. 523, 1950.

6. Truesdell, C., Correction and additions to "The mechanical foundations of elasticity and fluid dynamics". *Jour. Rat. Mech. Analysis* Vol. 2, p. 593, 1953.

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